CALCUL NUMERIC SEMINAR 4

NOTIȚE SUPORT SEMINAR

Cristian Rusu

EXEMPLU NUMERIC, EX. 1

- vectorii noștri inițiali sunt cu 3 elemente, deci d = 3
- transformarea ϕ are 9 elemente, deci D = 9

- transformările sunt:

$\phi(\mathbf{x}) =$ și $\phi(\mathbf{y}) =$

-

- $\phi(\mathbf{x})^T\phi(\mathbf{y}) =$
- $k(\mathbf{x}, \mathbf{y}) =$

EXEMPLU NUMERIC, EX. 1

- vectorii noștri inițiali sunt cu 3 elemente, deci d = 3
- transformarea ϕ are 9 elemente, deci D = 9

- transformările sunt:
- $$\phi(\mathbf{x}) = \begin{bmatrix} 4 \\ 6 \\ 8 \\ 6 \\ 9 \\ 12 \\ 8 \\ 12 \\ 16 \end{bmatrix}$$ și $\phi(\mathbf{y}) =$

- $\phi(\mathbf{x})^T \phi(\mathbf{y}) =$
- $k(\mathbf{x}, \mathbf{y}) =$

EXEMPLU NUMERIC, EX. 1

- vectorii noștri inițiali sunt cu 3 elemente, deci d = 3
- transformarea ϕ are 9 elemente, deci D = 9

- transformările sunt:
- $$\phi(\mathbf{x}) = \begin{bmatrix} 4 \\ 6 \\ 8 \\ 6 \\ 9 \\ 12 \\ 8 \\ 12 \\ 16 \end{bmatrix} \text{ și } \phi(\mathbf{y}) = \begin{bmatrix} 9 \\ 12 \\ 15 \\ 12 \\ 16 \\ 20 \\ 15 \\ 20 \\ 25 \end{bmatrix}$$

- $\phi(\mathbf{x})^T \phi(\mathbf{y}) =$
- $k(\mathbf{x}, \mathbf{y}) =$

EXEMPLU NUMERIC, EX. 1

- **vectorii noștri inițiali sunt cu 3 elemente, deci d = 3**
- **transformarea ϕ are 9 elemente, deci D = 9**

- **transformările sunt:**
- $$\phi(\mathbf{x}) = \begin{bmatrix} 4 \\ 6 \\ 8 \\ 6 \\ 9 \\ 12 \\ 8 \\ 12 \\ 16 \end{bmatrix} \text{ și } \phi(\mathbf{y}) = \begin{bmatrix} 9 \\ 12 \\ 15 \\ 12 \\ 16 \\ 20 \\ 15 \\ 20 \\ 25 \end{bmatrix}$$

- $\phi(\mathbf{x})^T\phi(\mathbf{y}) = 36 + 72 + 120 + 72 + 144 + 240 + 120 + 240 + 400 = 1444$
- $k(\mathbf{x}, \mathbf{y}) =$

EXEMPLU NUMERIC, EX. 1

- vectorii noștri inițiali sunt cu 3 elemente, deci d = 3
- transformarea ϕ are 9 elemente, deci D = 9

- transformările sunt:
- $\phi(\mathbf{x}) = \begin{bmatrix} 4 \\ 6 \\ 8 \\ 6 \\ 9 \\ 12 \\ 8 \\ 12 \\ 16 \end{bmatrix}$ și $\phi(\mathbf{y}) = \begin{bmatrix} 9 \\ 12 \\ 15 \\ 12 \\ 16 \\ 20 \\ 15 \\ 20 \\ 25 \end{bmatrix}$

- $\phi(\mathbf{x})^T \phi(\mathbf{y}) = 36 + 72 + 120 + 72 + 144 + 240 + 120 + 240 + 400 = 1444$
- $k(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^T \mathbf{y})^2 = (2 \times 3 + 3 \times 4 + 4 \times 5)^2 = (6 + 12 + 20)^2 = 38 \times 38 = 1444$

KERNEL POLINOMIAL, EX. 2

$k(\mathbf{x}, \mathbf{y}) =$

$=$

- $=$

$=$

- $\phi(z) =$

$\phi(\mathbf{x})^T \phi(\mathbf{y}) =$

-

KERNEL POLINOMIAL, EX. 2

$$k(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^T\mathbf{y} + 2)^2$$

$$=$$

- $$=$$

$$=$$

- $\phi(z) =$

$$\phi(\mathbf{x})^T\phi(\mathbf{y}) =$$

-

KERNEL POLINOMIAL, EX. 2

$$k(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^T \mathbf{y} + 2)^2$$

$$= ([x_1 x_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + 2)^2$$

- $=$

  $=$

- $\phi(z) =$

$\phi(\mathbf{x})^T \phi(\mathbf{y}) =$

-

KERNEL POLINOMIAL, EX. 2

$$k(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^T \mathbf{y} + 2)^2$$
$$= ([x_1 x_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + 2)^2$$

- $$= (x_1 y_1 + x_2 y_2 + 2)^2$$
$$=$$

- $\phi(z) =$

$\phi(\mathbf{x})^T \phi(\mathbf{y}) =$

-

KERNEL POLINOMIAL, EX. 2

$$k(\mathbf{x},\mathbf{y}) = (\mathbf{x}^T\mathbf{y} + 2)^2$$

$$= ([x_1 x_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + 2)^2$$

- $$= (x_1 y_1 + x_2 y_2 + 2)^2$$

$$= x_1^2 y_1^2 + x_2^2 y_2^2 + 2x_1 y_1 x_2 y_2 + 4x_1 y_1 + 4x_2 y_2 + 4$$

- $\phi(z) =$

$\phi(\mathbf{x})^T \phi(\mathbf{y}) =$

-

KERNEL POLINOMIAL, EX. 2

$$k(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^T \mathbf{y} + 2)^2$$

$$= ([x_1 x_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + 2)^2$$

- $$= (x_1 y_1 + x_2 y_2 + 2)^2$$

$$= x_1^2 y_1^2 + x_2^2 y_2^2 + 2 x_1 y_1 x_2 y_2 + 4 x_1 y_1 + 4 x_2 y_2 + 4$$

- $\phi(z) = \begin{bmatrix} 1 & z_1 & z_2 & z_1 z_2 & z_1^2 & z_2^2 \end{bmatrix}^T$

$\phi(\mathbf{x})^T \phi(\mathbf{y}) =$

-

KERNEL POLINOMIAL, EX. 2

$$k(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^T \mathbf{y} + 2)^2$$

$$= ([x_1 x_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + 2)^2$$

- $$= (x_1 y_1 + x_2 y_2 + 2)^2$$

$$= x_1^2 y_1^2 + x_2^2 y_2^2 + 2 x_1 y_1 x_2 y_2 + 4 x_1 y_1 + 4 x_2 y_2 + 4$$

- $\phi(z) = \begin{bmatrix} 1 & z_1 & z_2 & z_1 z_2 & z_1^2 & z_2^2 \end{bmatrix}^T$

- $$\phi(\mathbf{x})^T \phi(\mathbf{y}) = \begin{bmatrix} 1 & x_1 & x_2 & x_1 x_2 & x_1^2 & x_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ y_1 \\ y_2 \\ y_1 y_2 \\ y_1^2 \\ y_2^2 \end{bmatrix} = 1 + x_1 y_1 + x_2 y_2 + x_1 x_2 y_1 y_2 + x_1^2 y_1^2 + x_2^2 y_2^2$$

KERNEL RBF, EX. 3

$$\exp(-\gamma(x-y)^2) = \exp(-\gamma(x^2 - 2xy + y^2))$$
$$= \exp(-\gamma x^2 - \gamma y^2)\exp(2\gamma xy)$$
$$= \exp(-\gamma x^2 - \gamma y^2)\left(1 + \frac{2\gamma xy}{1!} + \frac{(2\gamma xy)^2}{2!} + \frac{(2\gamma xy)^3}{3!} + \dots\right)$$
$$= \exp(-\gamma x^2 - \gamma y^2)\left(1 \times 1 + \sqrt{\frac{2\gamma}{1!}}x \times \sqrt{\frac{2\gamma}{1!}}y\right.$$

- $$\left. + \sqrt{\frac{(2\gamma)^2}{2!}}x^2 \times \sqrt{\frac{(2\gamma)^2}{2!}}y^2 + \dots\right)$$

$$= \phi(\mathbf{x})^T\phi(\mathbf{y})$$

- $\phi(\mathbf{z}) =$

KERNEL RBF, EX. 3

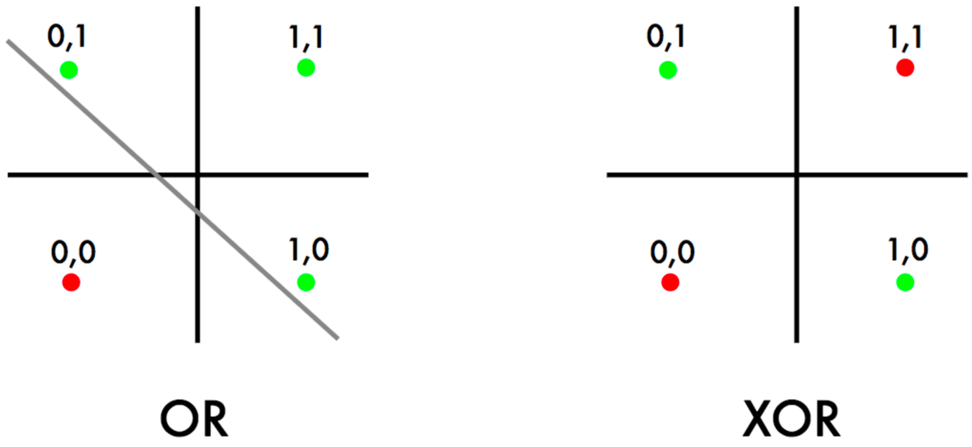
$$\exp(-\gamma(x-y)^2) = \exp(-\gamma(x^2 - 2xy + y^2))$$
$$= \exp(-\gamma x^2 - \gamma y^2)\exp(2\gamma xy)$$
$$= \exp(-\gamma x^2 - \gamma y^2)\left(1 + \frac{2\gamma xy}{1!} + \frac{(2\gamma xy)^2}{2!} + \frac{(2\gamma xy)^3}{3!} + \ldots\right)$$
$$= \exp(-\gamma x^2 - \gamma y^2)\left(1 \times 1 + \sqrt{\frac{2\gamma}{1!}}x \times \sqrt{\frac{2\gamma}{1!}}y\right.$$

- $$\left. + \sqrt{\frac{(2\gamma)^2}{2!}}x^2 \times \sqrt{\frac{(2\gamma)^2}{2!}}y^2 + \ldots\right)$$

$$= \phi(\mathbf{x})^T \phi(\mathbf{y})$$

- $$\phi(\mathbf{z}) = \exp(-\gamma z^2)\begin{bmatrix} 1 & \sqrt{\frac{2\gamma}{1!}}z & \sqrt{\frac{(2\gamma)^2}{2!}}z^2 & \sqrt{\frac{(2\gamma)^3}{3!}}z^3 & \ldots \end{bmatrix}^T$$

XOR PROBLEM, EX. 4

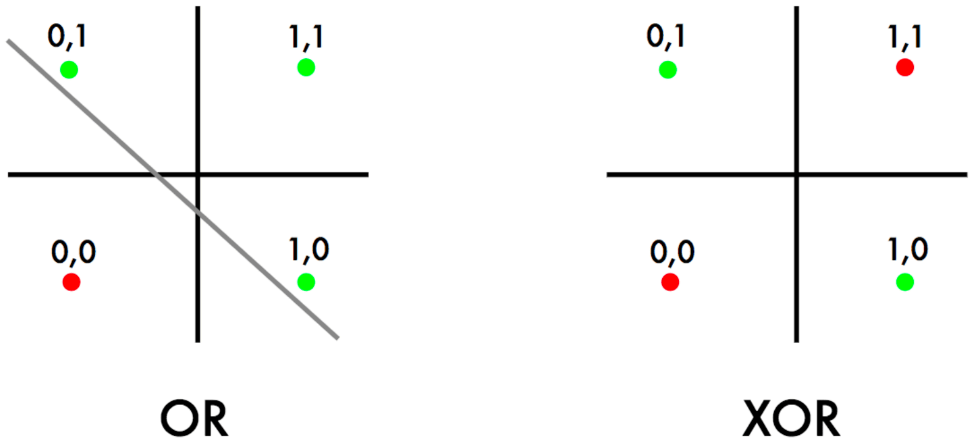


- **în loc de 0 și 1 putem să folosim -1 și 1 - problema este identică**

- **punctele sunt**

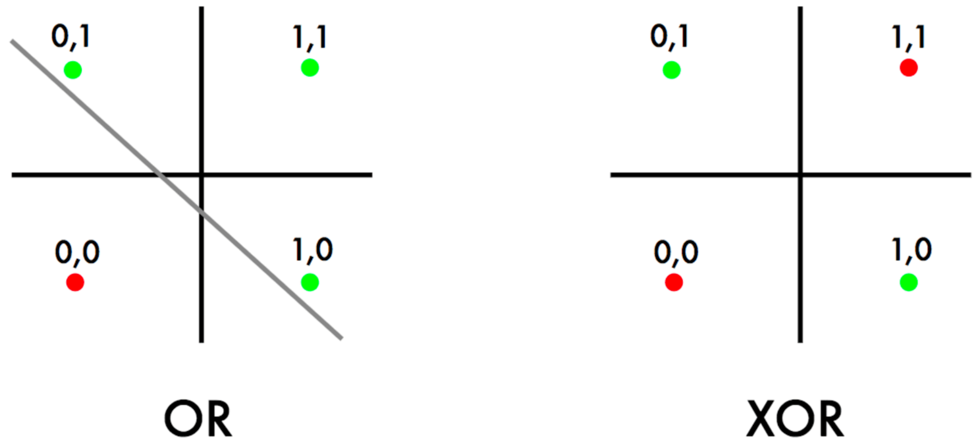
$$x_1 = [-1 \;\; -1]^T, y_1 = 1$$
$$x_2 = [-1 \;\; 1]^T, y_2 = -1$$
$$x_3 = [1 \;\; -1]^T, y_3 = -1$$
$$x_4 = [1 \;\; 1]^T, y_4 = 1$$

XOR PROBLEM, EX. 4



- **kernel-ul folosit este polinomial** $k(\mathbf{x}, \mathbf{y}) = (\mathbf{x}^T \mathbf{y} + 1)^2$
- $\phi(z) = \begin{bmatrix} 1 & \sqrt{2} z_1 & \sqrt{2} z_2 & z_1^2 & z_2^2 & \sqrt{2} z_1 z_2 \end{bmatrix}^T$
- **matricea de kernel este** $\mathbf{K} = \begin{bmatrix} 9 & 1 & 1 & 1 \\ 1 & 9 & 1 & 1 \\ 1 & 1 & 9 & 1 \\ 1 & 1 & 1 & 9 \end{bmatrix}$

XOR PROBLEM, EX. 4



- $$\phi(\mathbf{x}_1) = \begin{bmatrix} 1 \\ -\sqrt{2} \\ -\sqrt{2} \\ 1 \\ 1 \\ \sqrt{2} \end{bmatrix}, \phi(\mathbf{x}_2) = \begin{bmatrix} 1 \\ -\sqrt{2} \\ \sqrt{2} \\ 1 \\ 1 \\ -\sqrt{2} \end{bmatrix}, \phi(\mathbf{x}_3) = \begin{bmatrix} 1 \\ \sqrt{2} \\ -\sqrt{2} \\ 1 \\ 1 \\ -\sqrt{2} \end{bmatrix}, \phi(\mathbf{x}_4) = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{2} \\ 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$$

XOR PROBLEM, EX. 4

- cum pot să știu dacă în spațiul D = 6 dimensional exită o linie care separă cele două grupuri?

- trebuie să existe un w astfel încât
  $\mathbf{w}^T \phi(\mathbf{x}_i) > 0$ dacă $y_i = 1$ și $\mathbf{w}^T \phi(\mathbf{x}_i) < 0$ dacă $y_i = -1$

- rezultă

- $$\begin{bmatrix} 1 & -\sqrt{2} & -\sqrt{2} & 1 & 1 & \sqrt{2} \\ -1 & \sqrt{2} & -\sqrt{2} & -1 & -1 & \sqrt{2} \\ -1 & \sqrt{2} & \sqrt{2} & -1 & -1 & \sqrt{2} \\ 1 & \sqrt{2} & \sqrt{2} & 1 & 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

XOR PROBLEM, EX. 4

- cum pot să știu dacă în spațiul D = 6 dimensional exită o linie care separă cele două grupuri?

- trebuie să existe un w astfel încât $\mathbf{w}^T\phi(\mathbf{x}_i) > 0$ dacă $y_i = 1$ și $\mathbf{w}^T\phi(\mathbf{x}_i) < 0$ dacă $y_i = -1$

- rezultă

- $$\begin{bmatrix} 1 & -\sqrt{2} & -\sqrt{2} & 1 & 1 & \sqrt{2} \\ -1 & \sqrt{2} & -\sqrt{2} & -1 & -1 & \sqrt{2} \\ -1 & \sqrt{2} & \sqrt{2} & -1 & -1 & \sqrt{2} \\ 1 & \sqrt{2} & \sqrt{2} & 1 & 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- soluția w există dacă matricea de mai sus are rang 4, verificați

BIBLIOGRAFIE

- **aceste exerciții sunt bazate pe:**

  - https://mr-pc.org/t/cse5526/pdf/05c-svmKernels.pdf
  - https://www.ini.rub.de/PEOPLE/wiskott/Teaching/Material/KernelTrick-SolutionsPublic.pdf
  - https://www.cs.toronto.edu/~urtasun/courses/CSC411_Fall16/16_svm.pdf
  - https://classes.cec.wustl.edu/~SEAS-SVC-CSE517A/sp20/lecturenotes/09_lecturenote_kernels.pdf
  - https://www.csie.ntu.edu.tw/~cjlin/talks/kuleuven_svm.pdf
  - https://www.cs.utexas.edu/~dana/MLClass/XOR.pdf
  - https://dev.to/jbahire/demystifying-the-xor-problem-1blk
  - http://lcsl.mit.edu/courses/mlcc/mlcc2019/
  - http://www.cs.cornell.edu/courses/cs4780/2018fa/lectures/lecturenote13.html